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we are, of course, looking at the matter merely from a scientific point of view. Of course, most Christians believe that long before the natural life of this earth would be concluded, it will suffer a final catastrophe which will at least close the history of the human race on it as it exists now. Such catastrophes may, of course, occur to any planet by natural as well as supernatural causes; by collision with some other body, for instance; or to the whole planetary system, by some large body striking on the sun. One thing which we may perhaps look forward to is a time when, after the death or destruction of all the planets, the sun itself ceasing to be a luminary and furnace for bodies circulating round it, may itself become the great seat and home of life. In theorizing on this point we have no past experience or history to guide us. We shall see as we go on to discuss the stellar systems that we have at least one case, perhaps more than one, of a body sun-like in dimensions, which has either ceased to give light, or never gave it. It is only in exceptional cases that we have any means of recognizing the existence of such bodies; they may be very numerous. Neither can we tell whether the other innumerable brilliant suns scattered through space have attendant planets like our own. But it would be strange if they had not. If any considerable proportion of them have, evidently the chance that there are other habitable worlds in the universe becomes very great.

# ON HYPERBO-ELLIPTIC FUNCTIONS.

By Irving Stringham, Ph. D.

## I.

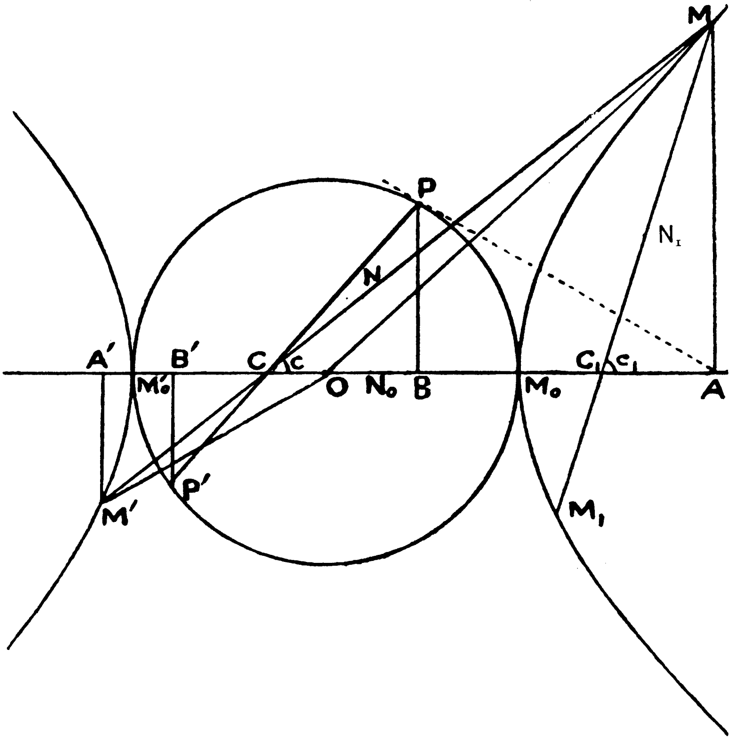
The geometrical definition of the Legendrian, or cyclo-elliptic functions by means of a circle, as given by the late M. Halphen in his *Traité des Fonctions Elliptiques*, t. I, p. 2, suggests an analogous construction for the equilateral hyperbola, through which what I venture to call the *hyperbo-elliptic functions* may in like manner be defined. M. Halphen's treatment is strictly geometrical, but the analytical form of presentation is better adapted to the analogous theory here sketched in outline.

## GEOMETRICAL REPRESENTATION.

With $M_0'M_0$ as principal diameter, construct an equilateral hyperbola, as in the accompanying figure, and let

$$\overline{M_0'M_0} = 2\,R = 2\,\overline{OM_0}.$$

Through a fixed point $C$ on the diameter draw the variable chord of the hyperbola $\overline{M'M}$ and from $M$ and $M'$ drop $\overline{MA}$ and $\overline{M'A'}$ perpendicular to $\overline{OA}$. Then we may write



$$\overline{OA} = R\cosh 2\phi, \qquad \overline{A\,M} = R\sinh 2\phi,$$
$$\overline{OA'} = R\cosh 2\phi', \qquad \overline{A'M'} = R\sinh 2\phi',$$

whence

$$\textit{sectorial area } O M_0\, M = \tfrac{1}{2}\, R^2\, 2\,\phi = R^2\phi,$$
$$\textit{sectorial area } O M_0'\, M' = \tfrac{1}{2}\, R^2\, 2\,\phi' = R^2\phi'.$$

Writing $\overline{CO} = \delta$, the position of the point $N$ on the chord $MM^1$ is defined by the equation

$$(1) \qquad CN = l\left\{\frac{2\,(R+\delta)}{MM'}\right\}^{\frac{1}{2}}\left[\cosh 2(\phi+\phi')\right]^{\frac{3}{4}},$$

in which $l$ is an arbitrary constant. Let $N_0$ be the position of $N$ when $CM$ coincides with $CM_0$, and let the sectorial area $CN_0N$ be represented by $l^2 u$. Then since to every point $M$ on the hyperbola there corresponds a unique position of $N$ on the curve $N_0N$, $u$ is a function of $\phi$. Imitating the nomenclature and notation of the cyclo-elliptic functions, I call $\phi$ the hyperbolic amplitude of $u$ and write

$$\phi = \mathrm{hm}\ u.$$

This functional relation will be more explicitly determined by means of the differential equation

$$d\phi = \mathrm{hm}'\ u\, du\ ;$$

for if the angle $N_0CN$ be denoted by $c$, the differential of the sectorial area $CN_0N$ is

$$l^2\, du = \tfrac{1}{2}\, \overline{CN}^2\, dc,$$

and in order to find hm′ $u$ it is only necessary to find the values of $dc$, $MM'$ and $\cosh 2\ (\phi + \phi')$ in terms of $\phi$ and $d\phi$.

The following equations are evident from the figure:

$$\begin{aligned} MM' \sin c &= R\ (\sinh 2\phi + \sinh 2\phi') \\ &= 2\ R \sinh (\phi + \phi') \cosh (\phi - \phi'), \\ MM^1 \cos c &= R\ (\cosh 2\phi + \cosh 2\phi') \\ &= 2\ R \cosh (\phi + \phi') \cosh (\phi - \phi'), \end{aligned} \tag{2}$$

$$\begin{aligned} CM \sin c &= R \sinh 2\phi, \\ CM \cos c &= R \cosh 2\phi + \delta\ ; \end{aligned} \tag{3}$$

$$\therefore \quad \tan c = \tanh (\phi + \phi') = \frac{R \sinh 2\phi}{R \cosh 2\phi + \delta}. \tag{4}$$

From the last equation the other functions of $\phi + \phi'$, and also those of $\phi - \phi'$, are easily found to be

$$\sinh (\phi + \phi') = \frac{R \sinh 2\phi}{D}\ ;\ \cosh (\phi + \phi') = \frac{R \cosh 2\phi + \delta}{D}\ ;$$

$$\sinh (\phi - \phi') = \frac{\delta \sinh 2\phi}{D}\ ;\ \cosh (\phi - \phi') = \frac{\delta \cosh 2\phi + R}{D},$$

where $\qquad D = \sqrt{R^2 + 2R\delta \cosh 2\phi + \delta^2}.$

Differentiating the several terms of equations (4) we have

$$\sec^2 c\ dc = \mathrm{sech}^2 (\phi + \phi')\ d\ (\phi + \phi') = \frac{2\ R\ (R + \delta \cosh 2\phi)}{(R \cosh 2\phi + \delta)^2}\, d\phi.$$

But from the same equations is also easily obtained

$$\sec^2 c = 1 + \tanh^2 (\phi + \phi') = \frac{\cosh 2(\phi + \phi')}{\cosh^2 (\phi + \phi')}\ ;$$

hence $$dc = \frac{d\,(\phi + \phi')}{\cosh 2\,(\phi + \phi')}$$
$$= \frac{2\,R\,(R + \delta \cosh 2\,\phi) \cosh^2 (\phi + \phi')}{(R \cosh 2\,\phi + \delta)^2 \cosh 2\,(\phi + \phi')}\,d\phi$$
$$= \frac{2\,R\,(R + \delta \cosh 2\,\phi)}{D^2 \cosh 2\,(\phi + \phi')}\,d\phi\,;$$

and since equations (1) and (2) give

$$\overline{CN}^2 = l^2 \frac{2\,(R + \delta)\,[\cosh 2\,(\phi + \phi')]^{\frac{3}{2}}}{2\,R \cosh (\phi + \phi') \cosh (\phi - \phi')} \cos c$$

$$\therefore \quad \tfrac{1}{2}\,\overline{CN}^2\,dc = l^2 \frac{(R + \delta)\,(R + \delta \cosh 2\,\phi)}{D^2 \cosh (\phi - \phi')}\,d\phi$$
$$= l^2 \frac{R + \delta}{D}\,d\phi = l^2 \frac{d\phi}{\sqrt{k^2 \sinh^2 \phi + 1}},$$

where $k^2 = \frac{4\,R\delta}{(R + \delta)^2}$. Hence the differential equation connecting $u$ and $\phi$ is

(5) $$du = \frac{d\phi}{\sqrt{k^2 \sinh^2 \phi + 1}},$$

and $$\operatorname{hm}' u = \sqrt{k^2 \sinh^2 \phi + 1}.$$

When the fixed point $C$ lies upon the diameter $M_0' M_0$ *produced*, say at $C_1$ as represented in the figure, the chord $MM_1$ has its extremities upon the same branch of the hyperbola, the argument formerly represented by $\phi'$ is now to be regarded as negative, say $-\phi_1$, and the equation defining the locus of $N_1$ is

(6) $$C_1 N_1 = l \left\{ \frac{2\,(R + \delta_1)}{MM_1} \right\}^{\frac{1}{2}} [\cosh 2(\phi - \phi_1)]^{\frac{3}{4}},$$

where $$\delta_1 = OC_1.$$

The transformations that lead to the differential equation in $\phi$ and $u$ differ but little from those of the preceding case, and I merely write down the successive results:

$$MM_1 \sin c_1 = R\,(\sinh 2\,\phi + \sinh 2\,\phi_1)$$
$$= \quad 2\,R \sinh (\phi + \phi_1) \cosh (\phi - \phi_1),$$
$$MM_1 \cos c_1 = R\,(\cosh 2\,\phi_1 - \cosh 2\,\phi)$$
$$= -\,2\,R \sinh (\phi + \phi_1) \sinh (\phi - \phi_1)\,;$$
$$C_1 M_1 \sin c_1 = R \sinh 2\,\phi,$$
$$C_1 M_1 \cos c_1 = R \cosh 2\,\phi - \delta_1\,;$$
$$\therefore \quad \tan c_1 = -\operatorname{ctnh} (\phi - \phi_1) = \frac{R \sinh 2\,\phi}{R \cosh 2\,\phi - \delta_1}.$$

$$\sinh(\phi - \phi_1) = \frac{R \cosh 2\phi - \delta_1}{D_1}; \quad \cosh(\phi - \phi_1) = \frac{R \sinh 2\phi}{D_1};$$

$$\sinh(\phi + \phi_1) = \frac{\delta_1 \cosh 2\phi - R}{D_1}; \quad \cosh(\phi + \phi_1) = \frac{\delta_1 \sinh 2\phi}{D_1};$$

$$D_1 = \sqrt{-R^2 + 2R\delta_1 \cosh 2\phi - \delta_1^2}.$$

$$\tfrac{1}{2}\,\overline{C_1N_1}^2\, dc = l^2 \frac{R + \delta_1}{D_1}\, d\phi = l^2 \frac{d\phi}{\sqrt{k_1^2 \cosh^2 \phi - 1}},$$

where $k_1^2 = \frac{4R\delta_1}{(R + \delta_1)^2}$. Representing this differential area by $l^2du$ we have the differential equation

$$(7) \qquad du = \frac{d\phi}{\sqrt{k_1^2 \cosh^2 \phi - 1}}$$

Suppose a solution of this equation to be represented in symbolic form by

$$\phi = \text{co-hm}\; u,$$

then $\phi$ is to be regarded as the hyperbolic coamplitude of $u$,

$$d\phi = \text{co-hm}'\, u\; du$$

and
$$\text{co-hm}'\, u = \sqrt{k_1^2 \cosh^2 \phi - 1}.$$

## DEFINITIONS.

Given $u$ and $\phi$ as functions of each other through the differential equation

$$du = \frac{d\,\phi}{\sqrt{k^2 \sinh^2 \phi + 1}}$$

the hyperbo-elliptic functions are defined as follows:

$$\phi = \text{hm}\; u, \text{ (hyperbolic amplitude of } u\text{)},$$
$$\sinh \phi = \text{hs}\; u,$$
$$\cosh \phi = \text{hc}\; u,$$
$$\sqrt{k^2 \sinh^2 \phi + 1} = \text{hd}\; u;$$

or, by means of the figure, they are defined geometrically as follows:

$$\text{hs}\; u = \frac{AM + A'M'}{MM'} \sqrt{\sec 2c},$$

$$\text{hc}\; u = \frac{AA'}{MM'} \sqrt{\sec 2c},$$

$$\text{hd}^2\, u = \left\{\frac{R - \delta}{R + \delta}\right\}^2 + \frac{4R\delta}{(R + \delta)^2} \left\{\frac{AA'}{MM'}\right\}^2 \sec 2c,$$

where $$\sec 2c = \frac{(MM')^2}{(AA')^2 - (AM + A'M')^2}.$$

The above expression for $\text{hd}^2\, u$ is readily obtained as the equivalent of $1 + k^2 \sinh^2 \phi$.

The following formulæ are immediate consequences of these definitions :

$$\begin{aligned} \text{hc}^2\, u - \text{hs}^2\, u &= 1, \\ \text{hd}^2\, u - k^2\, \text{hs}^2\, u &= 1\,; \end{aligned}$$

$$\begin{aligned} d\, \text{hm}\; u &= \text{hd}\; u\,.\,du, \\ d\, \text{hs}\;\; u &= \text{hc}\; u\,.\,\text{hd}\; u\,.\,du, \\ d\, \text{hc}\;\; u &= \text{hs}\; u\,.\,\text{hd}\; u\,.\,du, \\ d\, \text{hd}\;\; u &= k^2\, \text{hs}\; u\,.\,\text{hc}\; u\,.\,du. \end{aligned}$$

It is evident that hc $u$ and hd $u$ are never numerically less than 1 and that

$$\begin{aligned} \text{hm}\; 0 &= 0, & \text{hc}\; 0 &= 1, \\ \text{hs}\;\; 0 &= 0, & \text{hd}\; 0 &= 1. \end{aligned}$$

These equations are the analogues, line for line, of the following definitions and formulæ of the cyclo-elliptic functions, which I write down for comparison and reference.

If $$du = \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

then by definition

$$\begin{aligned} \theta &= \text{am}\; u, \text{ (amplitude of } u), \\ \sin\theta &= \text{sn}\;\; u, \\ \cos\theta &= \text{cn}\;\; u, \\ \sqrt{1 - \text{k}^2 \sin^2\theta} &= \text{dn}\;\; u\,; \end{aligned}$$

or, geometrically, by means of the figure, these definitions are

$$\text{sn}\; u = \frac{PB + P'B'}{PP'},$$

$$\text{cn}\; u = \frac{BB'}{PP'},$$

$$\text{dn}^2\, u = \left\{\frac{R + \delta}{R - \delta}\right\}^2 + \frac{4R\delta}{(R + \delta)^2}\left\{\frac{BB'}{PP'}\right\}^2;$$

while $2\theta =$ angle $POB$, $2\theta' =$ angle $P'OB'$, $\theta + \theta' =$ angle $PCB$ and the locus of the point $N$, on the chord $PP'$, is defined by

$$CN = l\left\{\frac{2(R + \delta)}{PP'}\right\}^{\frac{1}{2}}.$$

Then
$$\begin{aligned}\operatorname{sn}^2 u + \operatorname{cn}^2 u &= 1,\\ \operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u &= 1\,;\end{aligned}$$

and
$$\begin{aligned}d \operatorname{am} u &= \operatorname{dn} u \,.\, du,\\ d \operatorname{sn} u &= \operatorname{cn} u \,.\, \operatorname{dn} u \,.\, du,\\ d \operatorname{cn} u &= -\operatorname{sn} u \,.\, \operatorname{dn} u \,.\, du,\\ d \operatorname{dn} u &= -k^2 \operatorname{sn} u \,.\, \operatorname{cn} u \,.\, du.\end{aligned}$$

## RELATIONS BETWEEN HYPERBO-ELLIPTIC AND CYCLO-ELLIPTIC FUNCTIONS.

If in the differential expression $d\phi/\sqrt{1 - k^2 \sin^2 \phi}$ the substitution $\phi = -i\chi$ be made, the result is

$$du = \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{1}{i} \cdot \frac{d\chi}{\sqrt{k^2 \sinh^2 \chi + 1}}\,;$$

hence

$$d\,iu = \frac{d\chi}{\sqrt{k^2 \sinh^2 \chi + 1}} = \frac{d\,i\phi}{\sqrt{k^2 \sinh^2 i\phi + 1}},$$

and therefore

$$\begin{aligned}i\phi &= \operatorname{hm} iu = i \operatorname{am} u,\\ \operatorname{hs} iu &= \sinh i\phi = i \sin \phi = i \operatorname{sn} u,\\ \operatorname{hc} iu &= \cosh i\phi = \cos \phi = \operatorname{cn} u,\\ \operatorname{hd} iu &= \sqrt{-k^2 \operatorname{sn}^2 u + 1} = \sqrt{1 - k^2 \sin^2 \phi} = \operatorname{dn} u\,;\end{aligned}$$

or, changing $u$ into $iu$, these become

$$\begin{aligned}i \operatorname{am} iu &= \operatorname{hm} (-u) = -\operatorname{hm} u,\\ i \operatorname{sn} iu &= \operatorname{hs} (-u) = -\operatorname{hs} u,\\ \operatorname{cn} iu &= \operatorname{hc} (-u) = \operatorname{hc} u,\\ \operatorname{dn} iu &= \operatorname{dn} (-u) = \operatorname{dn} u,\end{aligned}$$

If in the same differential expression the substitution $\phi = \frac{\pi}{2} - i\psi$ be made, the result is

$$du = \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{d\psi}{\sqrt{k^2 \cosh^2 \psi - 1}},$$

and the following relations subsist between the functions of $\phi$ and $\psi$:

$$\begin{aligned}\cosh \psi &= \sin \phi = \operatorname{sn} u,\\ i \sinh \psi &= \cos \phi = \operatorname{cn} u,\\ \sqrt{1 - k^2 \cosh^2 \psi} &= \sqrt{1 - k^2 \sin^2 \phi} = \operatorname{dn} u.\end{aligned}$$

## ADDITION FORMULÆ.

I append the addition formulæ for the hyperbo-elliptic functions and a few others immediately deducible from them. These may be obtained from the corresponding formulæ of cyclo-elliptic functions by changing the argument $u$ into $iu$, or they may be deduced independently by integrating the equation

$$dx / \sqrt{(1 + x^2)(1 + k^2 x^2)} = dy / \sqrt{(1 + y^2)(1 + k^2 y^2)}.$$

The addition formulæ are

$$\mathrm{hs}\,(u \pm v) = \frac{\mathrm{hs}\,u.\mathrm{hc}\,v.\mathrm{hd}\,v \pm \mathrm{hc}\,u.\mathrm{hs}\,v.\mathrm{hd}\,u}{1 - k^2\,\mathrm{hs}^2\,u\,\mathrm{hs}^2\,v},$$

$$\mathrm{hc}\,(u \pm v) = \frac{\mathrm{hc}\,u.\mathrm{hc}\,v \pm \mathrm{hs}\,u.\mathrm{hs}\,v.\mathrm{hd}\,u.\mathrm{hd}\,v}{1 - k^2\,\mathrm{hs}^2\,u\,\mathrm{hs}^2\,v},$$

$$\mathrm{hd}\,(u \pm v) = \frac{\mathrm{hd}\,u.\mathrm{hd}\,v \pm k^2\,\mathrm{hs}\,u.\mathrm{hs}\,v.\mathrm{hc}\,u.\mathrm{hc}\,v}{1 - k^2\,\mathrm{hs}^2\,u\,\mathrm{hs}^2\,v}.$$

From the last two of these is obtained also

$$\mathrm{hc}\,(u \pm v) = \mathrm{hc}\,u.\mathrm{hc}\,v \pm \mathrm{hs}\,u.\mathrm{hs}\,v.\mathrm{hd}\,(u \pm v).$$

The following are immediate consequences of the foregoing:

$$\mathrm{hs}\,(u + v) + \mathrm{hs}\,(u - v) = \frac{2\,\mathrm{hs}\,u.\mathrm{hc}\,v.\mathrm{hd}.\,v}{1 - k^2\,\mathrm{hs}^2\,u\,\mathrm{hs}^2\,v},$$

$$\mathrm{hs}\,(u + v) - \mathrm{hs}\,(u - v) = \frac{2\,\mathrm{hc}\,u.\mathrm{hs}\,v.\mathrm{hd}\,u}{1 - k^2\,\mathrm{hs}^2\,u\,\mathrm{hs}^2\,v},$$

$$\mathrm{hs}\,(u + v) \cdot \mathrm{hs}\,(u - v) = \frac{\mathrm{hs}^2\,u - \mathrm{hs}^2\,v}{1 - k^2\,\mathrm{hs}^2\,u\,\mathrm{hs}^2\,v};$$

$$\mathrm{hs}\,2\,u = \frac{2\,\mathrm{hs}\,u.\mathrm{hc}\,u.\mathrm{hd}\,u}{1 - k^2\,\mathrm{hs}^4\,u},$$

$$\mathrm{hc}\,2\,u = \frac{\mathrm{hc}^2\,u + \mathrm{hs}^2\,u.\mathrm{hd}^2\,u}{1 - k^2\,\mathrm{hs}^4\,u},$$

$$\mathrm{hd}\,2\,u = \frac{\mathrm{hd}^2\,u + k^2\,\mathrm{hs}^2\,u.\mathrm{hc}^2\,u}{1 - k^2\,\mathrm{hs}^4\,u};$$

$$\mathrm{hc}\,2\,u - 1 = \frac{2\,\mathrm{hs}^2\,u.\mathrm{hd}^2\,u}{1 - k^2\,\mathrm{hs}^4\,u},$$

$$\mathrm{hc}\,2\,u + 1 = \frac{2\,\mathrm{hc}^2\,u}{1 - k^2\,\mathrm{hs}^4\,u},$$

$$\mathrm{hd}\,2\,u - 1 = \frac{2\,k^2\,\mathrm{hs}^2\,u.\mathrm{hc}^2\,u}{1 - k^2\,\mathrm{hs}^4\,u},$$

$$\mathrm{hd}\,2\,u + 1 = \frac{2\,\mathrm{hd}^2\,u}{1 - k^2\,\mathrm{hs}^4\,u}.$$

## II.

## A CLASSIFICATION OF ELLIPTIC DIFFERENTIALS OF THE FIRST KIND.

I sketch, in outline, the classification of elliptic differentials of the first kind into cyclo-elliptic and hyperbo-elliptic groups. Cayley's beautiful transformation of the general differential expression $dv/\sqrt{a + 4bv + 6cv^2 + 4dv^3 + ev^4}$ leads directly to such a classification. I write down Cayley's results without the proofs, referring the reader to the original paper in the *Cambridge and Dublin Mathematical Journal*, vol. I, (1846), pp. 70–73, reprinted in the *Collected Mathematical Papers*, vol. I, pp. 224–227. See also Cayley's *Elliptic Functions*, pp. 317–320.

The investigation shows that if

$$V \equiv a + 4bv + 6cv^2 + 4dv^3 + ev^4$$

be transformed into

$$V' \equiv a'(1 + pv'^2)(1 + qv'^2)$$

by the substitution

$$v = \frac{\lambda + \mu v'}{\lambda_1 + \mu_1 v'}$$

then

$$\frac{dv}{\sqrt{V}} = \left\{\frac{p^2 + q^2 + 14pq}{12I}\right\}^{\frac{1}{4}} \frac{\sqrt{a'}\,dv'}{\sqrt{V'}}$$

where

$$I \equiv ae - 4bd + 3c^2,$$

and that if

$$J \equiv ace + 2bcd - ad^2 - eb^2 - c^3,$$

the equation connecting $p$ and $q$ is

$$108pq(p - q)^4 I^3 = (p^2 + q^2 + 14pq)^3 \Delta,$$

where $\Delta \equiv I^3 - 27J^2$; and finally, that this equation of the sixth degree in $q/p$ or $p/q$ yields, when the discriminant of $V$ is positive, four positive real and two imaginary roots, but when the discriminant is negative, two roots real and negative and four roots imaginary.

Attending to real values only of this ratio, it follows that $p$ and $q$ have the same or different signs according as the discriminant $\Delta$ is positive or negative; or this may be seen directly from the equation in $p$ and $q$, which, since

$$(\lambda\mu_1 - \lambda_1\mu)^4 I = I' \equiv \frac{a'^2}{12}(p^2 + q^2 + 14pq),$$

may be written in the form

$$108\, p\, q\, (p - q)^4\, a'^6 = (\lambda\, \mu_1 - \lambda_1\, \mu)^{12}\, \Delta,$$

whence it appears that the product $p\, q$ must have the same sign as $\Delta$. Hence, we may make the following classification, wherein $p$ and $q$ are now supposed to be positive in all cases, and

$$H \equiv a\, c - b^2,$$

$$I \equiv a e - 4\, b\, d + 3\, c^2,$$

$$J \equiv a\, c\, e + 2\, b\, c\, d - a\, d^2 - e\, b^2 - c^3,$$

$$\Delta \equiv I^3 - 27\, J^2.$$

*A*. If the discriminant of $V$ be positive, that is, if $V$ be composed of either four real, or four imaginary factors, the coefficients of $v'^2$ are either both positive, or both negative.

(*i*). If $\Delta > 0$ and both the conditions

$$\mathrm{H} < 0, \quad a^2\, I - 12\, H^2 < 0\,^*$$

be satisfied, all four factors are real, the coefficients of $v'^2$ are both negative, and

$$\frac{d\, v}{\sqrt{V}} = \left\{\frac{p^2 + q^2 + 14\, p\, q}{12\, I}\right\}^{\frac{1}{4}} \frac{d\, v'}{\sqrt{(1 - p\, v'^2)\, (1 - q\, v'^2)}}$$

(*ii*). But if $\Delta > 0$ and either

$$\mathrm{H} > 0, \quad \text{or} \quad a^2\, I - 12\, H^2 > 0,^*$$

all the factors are imaginary, the coefficients of $v'^2$ are both positive, and

$$\frac{d\, v}{\sqrt{V}} = \left\{\frac{p^2 + q^2 + 14\, p\, q}{12\, I}\right\}^{\frac{1}{4}} \frac{d\, v'}{\sqrt{(1 + p\, v'^2)\, (1 + q\, v'^2)}}.$$

*B*. If the discriminant of $V$ be negative, that is, if $V$ be composed of two real and two imaginary factors, then one of the coefficients of $v'^2$ is positive and the other negative; hence,

if $\Delta < 0$

$$\frac{d\, v}{\sqrt{V}} = \left\{\frac{p^2 + q^2 - 14\, p\, q}{12\, I}\right\}^{\frac{1}{4}} \frac{d\, v'}{\sqrt{(1 \pm p\, v'^2)\, (1 \mp q\, v'^2)}}$$

Each of these three forms, as is well known, is reducible to either of the others by a quadric transformation. Thus the third, by means of the substitution

$$v'^2 = \frac{w^2}{1 \mp p\, w^2}$$

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* Burnside and Panton's *Theory of Equations*, p. 143.

becomes

$$\frac{dv}{\sqrt{V}} = \left\{\frac{p^2 + q^2 - 14\,p\,q}{12\,I}\right\}^{\frac{1}{4}} \frac{dw}{\sqrt{1 \mp p\,w^2 \,.\, 1 \mp (p + q)\,w^2}}$$

and, by a proper choice of signs, the same form of substitution will transform the first or second into one of the others.

The final reductions are accomplished by familiar transformations. Making $q\,/p$, or $p\,/q = k^2$, according as $p$ is greater or less than $q$, and letting $p\,v'^2$, or $q\,v'^2 = z^2$, in accordance with the same conditions respectively, the three differential expressions become

$$(i)\quad \frac{dv}{\sqrt{V}} = \left\{\frac{k^2 + 14\,k + 1}{12\,I}\right\}^{\frac{1}{4}} \frac{dz}{\sqrt{(1 - z^2)\,(1 - k^2\,z^2)}},$$

$$(ii)\quad \frac{dv}{\sqrt{V}} = \left\{\frac{k^2 + 14\,k + 1}{12\,I}\right\}^{\frac{1}{4}} \frac{dz}{\sqrt{(1 + z^2)\,(1 + k^2\,z^2)}},$$

$$(iii)\quad \frac{dv}{\sqrt{V}} = \left\{\frac{k^2 - 14\,k + 1}{12\,I}\right\}^{\frac{1}{4}} \frac{dz}{\sqrt{(1 \pm z^2)\,(1 \mp k^2\,z^2)}}.$$

If we suppose the given differential to be reduced to either the first or second form, the final substitutions to be made are as follows:

(*i*). *In the first form:* when $z^2 < 1$, the proper substitution is $z = \sin\phi$, and we have the ordinary case

$$\frac{dz}{\sqrt{(1 - z^2)\,(1 - k^2\,z^2)}} = \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}.$$

When $z^2 > \frac{1}{k^2}$, $\phi$ is complex and may be taken $= \frac{\pi}{2} - i\,\psi$; the substitution is $z = \sin\phi = \cosh\psi$, and the differential takes the form

$$\frac{dz}{\sqrt{(z^2 - 1)\,(k^2\,z^2 - 1)}} = \frac{d\psi}{\sqrt{k^2 \cosh^2\psi - 1}}.$$

When $z^2$ lies between 1 and $\frac{1}{k^2}$ the denominator, if it appear in the form $\sqrt{(1 - z^2)\,(1 - k^2\,z^2)}$, is imaginary, but we may consider the differential expression $dz\,/\sqrt{(z^2 - 1)\,(1 - k^2\,z^2)}$, and in this case make the substitution $z = \cosh\psi$, by which we obtain

$$\frac{dz}{\sqrt{(z^2 - 1)\,(1 - k^2\,z^2)}} = \frac{d\psi}{\sqrt{1 - k^2 \cosh^2\psi}}.$$

(*ii*). *In the second form* the substitution obviously is, for all values of $z$, $z = \sinh\chi$, and the result is

$$\frac{dz}{\sqrt{(z^2 + 1)\,(k^2\,z^2 + 1)}} = \frac{d\chi}{\sqrt{k \sinh^2\chi + 1}}.$$

BERKELEY, September, 1889.